

Effective Matter Cosmologies of Massive Gravity I: Non-Physical Fluids

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Abstract

For the massive gravity, after decoupling from the metric equation we find a broad class of solutions of the Stückelberg sector by solving the background metric in the presence of a diagonal physical metric. We then construct the dynamics of the corresponding FLRW cosmologies which inherit effective matter contribution through the decoupling solution mechanism of the scalar sector.

Keywords: Non-linear theories of gravity, massive gravity, Cosmological solutions

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1 Introduction

The Fierz-Pauli [1] massive gravity theory has been extended to a non-linear Boulware-Deser (BD) [2, 3] ghost-free level in [4, 5]. Later this theory is upgraded to include a general background metric [6, 7, 8]. The cosmological

solutions of this non-linear and ghost-free theory has been an active topic of research in recent years [9].

In [10] we have studied the Einstein solutions of the so-called minimal sector of the massive gravity. On the other hand [11] was devoted to develop a methodology again for the minimal theory which would exactly solve the Stückelberg sector by first constructing the solution generating background (fiducial) metric. The approach in both of these works was to construct a solution ansatz which would decouple the metric and the scalar sectors. This enables the determination of the background metric satisfying the ansatz constraint which leads to the solution of the scalar sector. In the following work we adopt a similar formalism for the most general massive gravity theory. Our objective will be to construct an ansatz which will function in the same direction. As a solution to this ansatz constraint we will determine the background metric which will lead us to a class of solutions of the scalar sector for a given diagonally-formed physical metric. Such a formulation will replace solving the scalars from the dynamical field equations by a semi-algebraic solution of the corresponding ansatz equation. As a physical consequence of such a solution method admitting decoupling of the field equations of the metric and the scalar sectors we will be able to construct the FLRW cosmological dynamics associating the above-mentioned scalar moduli and the background metric. We will show that the metric sector thus the cosmological equations get contributions from an effective matter energy-momentum tensor which parametrizes the ansatz we consider and which enters into the metric equation as a remainder of the act of decoupling the scalars from it. We will also discuss the conservation relation the effective matter must satisfy. This is a modified version of the usual energy-momentum conservation. Therefore the effective ideal fluid appearing in the cosmological equations must be considered as a non-physical one.

In Section two we construct the necessary ansatz mentioned above which will decouple the Stückelberg scalars from the metric sector by introducing an effective energy-momentum tensor. In Section three we will present the explicit solutions of the ansatz equation for the background metric and the scalars of the theory. We will also discuss the constraint equation to be satisfied by the effective matter so that the scalar solutions obtained become also the solutions of the theory. Finally Section four will be reserved for the discussion of the dynamics of the corresponding cosmological solutions of the general massive gravity which possess effective matter terms as modifications in the Friedmann and the cosmic acceleration equations.

2 The ansatz

The ghost-free massive gravity action with a general background metric which is coupled to matter can be given as [6]

$$S = -M_p^2 \int \left[R * 1 - 2m^2 \sum_{n=0}^3 \beta_n e_n(\sqrt{\Sigma}) * 1 + \Lambda * 1 \right] - S_{MATT}, \quad (2.1)$$

where β_n are arbitrary coefficients. M_p is the Planck mass, m is the graviton mass, R is the Ricci scalar, and Λ is the cosmological constant. Here $\{e_n(\sqrt{\Sigma})\}$ are the elementary symmetric polynomials

$$\begin{aligned} e_0 &\equiv e_0(\sqrt{\Sigma}) = 1, \\ e_1 &\equiv e_1(\sqrt{\Sigma}) = \text{tr} \sqrt{\Sigma}, \\ e_2 &\equiv e_2(\sqrt{\Sigma}) = \frac{1}{2} ((\text{tr} \sqrt{\Sigma})^2 - \text{tr}(\sqrt{\Sigma})^2), \\ e_3 &\equiv e_3(\sqrt{\Sigma}) = \frac{1}{6} ((\text{tr} \sqrt{\Sigma})^3 - 3 \text{tr} \sqrt{\Sigma} \text{tr}(\sqrt{\Sigma})^2 + 2 \text{tr}(\sqrt{\Sigma})^3), \end{aligned} \quad (2.2)$$

of the four by four matrix functional $\sqrt{\Sigma}$ in which

$$(\Sigma)^\mu{}_\nu = g^{\mu\rho} \partial_\rho \phi^a \partial_\nu \phi^b \bar{f}_{ab}(\phi^c), \quad (2.3)$$

with $g^{\mu\nu}$ being the inverse physical metric, $\{\phi^a(x^\mu)\}$ for $a = 0, 1, 2, 3$ the Stückelberg scalars, and $\bar{f}_{ab}(\phi^c)$ the background metric. The square root matrix is defined via $\sqrt{\Sigma} \sqrt{\Sigma} = \Sigma$. The metric equation corresponding to (2.1) can be derived as [6]

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \frac{1}{2} \Lambda g_{\mu\nu} + \mathcal{T}_{\mu\nu}^S = G_N T_{\mu\nu}^M, \quad (2.4)$$

where $T_{\mu\nu}^M$ is the physical matter energy-momentum tensor and for later convenience we have written compactly the contribution of the Stückelberg sector as $\mathcal{T}_{\mu\nu}^S$ which is derived in [6] as

$$\mathcal{T}_{\mu\nu}^S = \frac{1}{2} m^2 \left[\sum_{n=0}^3 (-1)^n \beta_n \left(g_{\mu\lambda} Y_{n\nu}^\lambda(\sqrt{\Sigma}) + g_{\nu\lambda} Y_{n\mu}^\lambda(\sqrt{\Sigma}) \right) \right], \quad (2.5)$$

where

$$\begin{aligned}
Y_0(\sqrt{\Sigma}) &= \mathbf{1}_4, \\
Y_1(\sqrt{\Sigma}) &= \sqrt{\Sigma} - \text{tr}\sqrt{\Sigma} \mathbf{1}_4, \\
Y_2(\sqrt{\Sigma}) &= (\sqrt{\Sigma})^2 - \text{tr}\sqrt{\Sigma}\sqrt{\Sigma} + \frac{1}{2}[(\text{tr}\sqrt{\Sigma})^2 - \text{tr}(\sqrt{\Sigma})^2] \mathbf{1}_4, \\
Y_3(\sqrt{\Sigma}) &= (\sqrt{\Sigma})^3 - \text{tr}\sqrt{\Sigma}(\sqrt{\Sigma})^2 + \frac{1}{2}[(\text{tr}\sqrt{\Sigma})^2 - \text{tr}(\sqrt{\Sigma})^2] \sqrt{\Sigma} \\
&\quad - \frac{1}{6}[(\text{tr}\sqrt{\Sigma})^3 - 3\text{tr}\sqrt{\Sigma}\text{tr}(\sqrt{\Sigma})^2 + 2\text{tr}(\sqrt{\Sigma})^3] \mathbf{1}_4,
\end{aligned} \tag{2.6}$$

with $\mathbf{1}_4$ being the four by four unit matrix. Now if we define the matrix $[\mathcal{T}^S]^\mu{}_\nu \equiv \mathcal{T}_{\mu\nu}^S$ then in matrix notation we can write (2.5) as

$$\mathcal{T}^S = \frac{1}{2}m^2 \left[\sum_{n=0}^3 (-1)^n \beta_n (gY_n + (gY_n)^T) \right]. \tag{2.7}$$

Now by using the symmetries [12]

$$g(\sqrt{\Sigma})^n = (g(\sqrt{\Sigma})^n)^T, \tag{2.8}$$

for any integer n and also by referring to the definitions of the symmetric polynomials (2.2) we can write (2.7) as

$$\begin{aligned}
\mathcal{T}^S &= m^2 \left[-\beta_1 g\sqrt{\Sigma} + \beta_2 g(\sqrt{\Sigma})^2 - \beta_2 \text{tr}\sqrt{\Sigma} g\sqrt{\Sigma} - \beta_3 g(\sqrt{\Sigma})^3 \right. \\
&\quad \left. + \beta_3 \text{tr}\sqrt{\Sigma} g(\sqrt{\Sigma})^2 - \frac{1}{2}\beta_3 ((\text{tr}\sqrt{\Sigma})^2 - \text{tr}(\sqrt{\Sigma})^2) g\sqrt{\Sigma} \right. \\
&\quad \left. + \left(\sum_{n=0}^3 \beta_n e_n \right) g \right].
\end{aligned} \tag{2.9}$$

To decouple the Stückelberg sector from the metric one and then to solve for the background metric and the scalars of the theory we now introduce the solution ansatz

$$\mathcal{T}^S = C_1 m^2 g + C_2 m^2 \tilde{T} - \frac{1}{2} C_2 m^2 \tilde{T}^\mu{}_\mu g + C_2 m^2 \tilde{\tau}, \tag{2.10}$$

where C_1, C_2 are arbitrary constants and in matrix sense $[\tilde{T}]^\mu{}_\nu \equiv \tilde{T}_{\mu\nu}$ is completely an arbitrary symmetric tensor parametrizing the solution moduli

for which later we will show that it enters into the metric sector as an effective energy-momentum tensor source. We have defined its trace as

$$\tilde{T}^\mu{}_\mu = \text{tr}(g^{-1}\tilde{T}) = g^{\mu\nu}\tilde{T}_{\mu\nu}. \quad (2.11)$$

In (2.10) we have also introduced the matrix $[\tilde{\tau}]^\mu{}_\nu \equiv \tilde{\tau}_{\mu\nu}$ with the definition

$$\tilde{\tau}_{\mu\nu} = g^{\alpha\beta} \frac{\delta \tilde{T}_{\alpha\beta}}{\delta g^{\mu\nu}}, \quad (2.12)$$

by assuming that the effective energy-momentum tensor will depend explicitly on the physical metric which will be the case for the cosmological solutions. Our fundamental motive in proposing this form of a solution ansatz has a cosmological perspective. The introduction of the first two terms in (2.10) is for designing solutions in which the massive sector of the theory contributes an effective cosmological constant and a non-matter originated energy-momentum tensor to the metric equations. In this way the cosmological equations of the theory can be formulated in the canonical form of the general relativity (GR) ones with additional effective cosmological constant and matter contribution which can work as a cure for the dark energy problem of GR cosmology by admitting self-accelerating solutions. However as we will discuss next the last two terms in (2.10) are needed for mathematical consistency so that (2.10) becomes a soluble ansatz. At this stage there is no guarantee that such an ansatz may lead to solutions however below we will show that it admits solutions both in the Stückelberg and the metric sectors thus it is a legitimate one. To generate solutions firstly we have to make the observation that on the right hand side in (2.9) there appears

$$\mathcal{L}_S = \sum_{n=0}^3 \beta_n e_n, \quad (2.13)$$

which is proportional to the Lagrangian of the Stückelberg coupling of the massive gravity action in (2.1). Now let us observe that via (2.1) and (2.4) we have the inverse-metric variation

$$\delta(2m^2\sqrt{-g} \sum_{n=0}^3 \beta_n e_n) = -\sqrt{-g} \mathcal{T}_{\mu\nu}^S \delta g^{\mu\nu}. \quad (2.14)$$

Upon the introduction of the solution ansatz (2.10) one can ask the question of what on-shell Lagrangian \mathcal{L}_{OS} this ansatz can be derived from, in other

words what the corresponding Lagrangian level ansatz is. (2.10) and (2.14) show that \mathcal{L}_{OS} must satisfy the inverse-metric variation

$$\begin{aligned} \delta(2m^2\sqrt{-g}\mathcal{L}_{OS}) = & \sqrt{-g}(-C_1m^2g_{\mu\nu} - C_2m^2\tilde{T}_{\mu\nu} - C_2m^2\tilde{\tau}_{\mu\nu} \\ & + \frac{1}{2}C_2m^2\tilde{T}^\rho{}_\rho g_{\mu\nu})\delta g^{\mu\nu}. \end{aligned} \quad (2.15)$$

Since for a general Lagrangian the variation solely with respect to the inverse metric leads to

$$\delta(\sqrt{-g}\mathcal{L}) = \sqrt{-g}\left(\frac{\delta\mathcal{L}}{\delta g^{\mu\nu}} - \frac{1}{2}\mathcal{L}g_{\mu\nu}\right)\delta g^{\mu\nu}, \quad (2.16)$$

we can deduce that

$$\mathcal{L}_{OS} = C_1 - \frac{1}{2}C_2\tilde{T}^\mu{}_\mu. \quad (2.17)$$

We conclude that the solutions which satisfy (2.10) at the Lagrangian level must lead to $\mathcal{L}_S = \mathcal{L}_{OS}$. Therefore for the symmetric polynomials of the matrix $\sqrt{\Sigma}$ we have the on-shell relation

$$\sum_{n=0}^3 \beta_n e_n = C_1 - \frac{1}{2}C_2\tilde{T}^\mu{}_\mu. \quad (2.18)$$

This analysis shows the necessity of adding the last two terms to the ansatz (2.10). If one plans to have the second generic term in (2.10) then (2.17) is the simplest¹ Lagrangian level ansatz that contains the effective energy-momentum tensor explicitly in it and produces the second term in (2.10) upon variation with respect to the inverse metric. However (2.17) also produces the last two terms of (2.10). Hence they must be included in the solution ansatz when one fixes the on-shell Lagrangian as (2.17) in its simplest form.

3 The Stückelberg Sector

In this section we will focus on solving the Stückelberg scalars and the background metric in the solution ansatz (2.10). First let us take the trace of

¹We did not check any other possible form of Lagrangian which contains explicitly the effective energy-momentum tensor in it and which gives such a term. On the other hand we should state that a simple form for the Lagrangian level ansatz is necessary for deriving diagonal solutions, and for the simple form of these solutions.

(2.10). If we multiply both sides by g and then take the trace of the matrix equation in (2.10) by also using (2.2) after some algebra we find

$$4\beta_0 e_0 + 3\beta_1 e_1 + 2\beta_2 e_2 + \beta_3 e_3 = 4C_1 - C_2 \tilde{T}^\mu{}_\mu + C_2 \tilde{\tau}^\mu{}_\mu, \quad (3.1)$$

where we introduce the trace

$$\tilde{\tau}^\mu{}_\mu = \text{tr}(g^{-1}\tilde{\tau}) = g^{\mu\nu}\tilde{\tau}_{\mu\nu}. \quad (3.2)$$

Using (2.18) in the above equation will eliminate e_3 and then we can solve for e_2 . The computation reads

$$e_2 = \frac{1}{\beta_2} \left[3C_1 - 3\beta_0 - 2\beta_1 e_1 - \frac{1}{2} C_2 \tilde{T}^\mu{}_\mu + C_2 \tilde{\tau}^\mu{}_\mu \right], \quad (3.3)$$

where we have also used $e_0 = 1$. Now that we have found e_2 in terms of $e_1 = \text{tr}\sqrt{\Sigma}$ we can turn our attention to finding solutions to the ansatz (2.10). Substituting (2.9) in (2.10) then multiplying both sides by g and using (2.18)² lead us to the cubic matrix equation for $\sqrt{\Sigma}$

$$A(\sqrt{\Sigma})^3 + B(\sqrt{\Sigma})^2 + C(\sqrt{\Sigma}) + D = 0, \quad (3.4)$$

where we have introduced the four by four matrices

$$\begin{aligned} A &= -\beta_3 \mathbf{1}_4, \\ B &= (\beta_2 + \beta_3 e_1) \mathbf{1}_4, \\ C &= (-\beta_1 - \beta_2 e_1 - \beta_3 e_2) \mathbf{1}_4, \\ D &= -C_2 g^{-1} \tilde{T} - C_2 g^{-1} \tilde{\tau}. \end{aligned} \quad (3.5)$$

Although on-shell we have derived e_2 in terms of e_1 which is not specified yet we will keep the compact notation of e_2 in the following formulation for the sake of simple appearance of the equations. To be able to solve (3.4) we will assume that the physical metric g , Σ (thus $\sqrt{\Sigma}$), the effective energy-momentum tensor \tilde{T} , and $\tilde{\tau}$ are all diagonal matrices so that (3.4) becomes a diagonal matrix equation. Our results in the following analysis will justify

²This substitution of the on-shell Lagrangian in (3.1) and here enables us to eliminate first e_3 , then e_2 and to obtain the matrix equation (3.4) which promises simple solutions without the presence of complicated trace coefficients. Also the simple form of (2.17) contributes to the simplicity of the D - term in (3.4) which will support our diagonality assumption to generate solutions in the following analysis.

that there exist diagonal Σ solutions to (3.4) and we are free to specify the form of \tilde{T} and we restrict ourselves to the diagonal metric solutions of the metric sector. This scheme is also conformal with the cosmological solutions we will consider later. We should state here that the solution ansatz (2.10) is independent of the diagonality assumption we will choose for the equation (3.4) in the following analysis. However for a non-diagonal choice of $\sqrt{\Sigma}$ equation (3.4) will give a set of coupled cubic algebraic equations for the entries of the matrix $\sqrt{\Sigma}$. Also when solved from (3.4) the non-diagonal form of $\sqrt{\Sigma}$ will lead us to a set of coupled first-order partial differential equations. In this more general picture of solutions one can also choose non-diagonal physical metrics g , and the tensors \tilde{T} , $\tilde{\tau}$. Thus more general set of non-diagonal solutions to (3.4) can be derived by choosing various non-diagonal solution forms of $\sqrt{\Sigma}$, g , and \tilde{T} but in this case one has to face the difficulty of algebraic and later differential coupling of equations. On the other hand the diagonality assumption of the ingredients of (3.4) decouples the algebraic equations for the entries of $\sqrt{\Sigma}$ first and then the partial differential equations for the Stückelberg fields later. As a final remark in this direction: assuming diagonality for $\sqrt{\Sigma}$ but not for g will put extra constraints on the effective energy-momentum tensor \tilde{T} via (3.4). Now let us define

$$\begin{aligned}\Delta_0 &= B^2 - 3AC, \\ \Delta_1 &= 2B^3 - 9ABC + 27A^2D, \\ \Delta &= -\frac{1}{27}(A^2)^{-1}[\Delta_1^2 - 4\Delta_0^3].\end{aligned}\tag{3.6}$$

If $\Delta > 0$ (3.4) has three distinct real roots, if $\Delta < 0$ then it has two complex and one real root, also if $\Delta = 0$ then there are three real roots again with a two-fold degeneracy. The general solutions of (3.4) can be given as

$$\sqrt{\Sigma} = \mathcal{G}_i, \quad \text{for } i = 1, 2, 3,\tag{3.7}$$

where

$$\mathcal{G}_i = -\frac{1}{3}A^{-1}[B\mathbf{1}_4 + u_i\mathcal{U} + u_i^{-1}\mathcal{U}^{-1}\Delta_0],\tag{3.8}$$

with

$$u_1 = 1, \quad u_2 = \frac{1}{2}(-1 + i\sqrt{3}), \quad u_3 = \frac{1}{2}(-1 - i\sqrt{3}),\tag{3.9}$$

and

$$\mathcal{U} = \left[\frac{\Delta_1 + \sqrt{\Delta_1^2 - 4\Delta_0^3}}{2} \right]^{1/3}.\tag{3.10}$$

For the general solutions (3.7) of (3.4) the elementary symmetric polynomial function e_1 namely $tr\sqrt{\Sigma}$ is left undetermined as a solution parametrizing function. The basic reason for this degree of freedom in the solution generating scheme we have constructed is the following: originally there exist two independent parameters namely e_2 , and e_1 in the solution ansatz (2.10) as we can eliminate e_3 by the introduction of the on-shell Lagrangian (2.18), when one takes the trace of (2.10) which is equivalent to (3.4) one obtains (3.3) which enables us to express e_2 in terms of e_1 , on the other hand one may expect to determine e_1 from the explicit solution (3.7) by taking the trace of it as well, however (3.7) is algebraically another way of writing (3.4) which it satisfies as a solution thus taking its trace would lead us to no additional identity than (3.3). Therefore we can conclude that in the solution (3.7) e_1 remains as a free function to parametrize the solutions. Below we will see that demanding the reality of the solutions may bring restrictions on e_1 or fix and determine its functional form in terms of the other free function parameters of the solutions (i.e. the effective energy-momentum tensor). Now by assuming that we focus on the real solutions of (3.4) squaring both sides of (3.7) yields

$$\Sigma = g^{-1}f = \mathcal{G}^2, \quad (3.11)$$

where we introduce the pull-back of the background metric \bar{f} by using the Stückelberg co-ordinate transformations $\{\phi^a(x^\mu)\}$ as

$$[f]^\mu{}_\nu \equiv f_{\mu\nu} = \partial_\mu \phi^a \partial_\nu \phi^b \bar{f}_{ab}(\phi^c). \quad (3.12)$$

Defining $\mathcal{G}' = g\mathcal{G}^2$ with $\mathcal{G}'_{\mu\nu} \equiv [\mathcal{G}']^\mu{}_\nu$ we can write (3.11) as

$$\partial_\mu \phi^a \partial_\nu \phi^b \bar{f}_{ab}(\phi^c) = \mathcal{G}'_{\mu\nu}. \quad (3.13)$$

From this equation we deduce that f must be a diagonal matrix as also it was obvious from our construction. Now let us also assume that the background metric \bar{f} is also of the diagonal form. To find solutions to these set of equations we will follow the method developed in [10, 11]. Firstly we choose \bar{f} as

$$\bar{f} = \text{diag}(f_{00}, f_{ii}), \quad \text{for } i = 1, 2, 3. \quad (3.14)$$

With this choice therein and bearing in mind that the right hand side is a diagonal matrix, from (3.13) we obtain

$$\sum_{a=0}^3 (\partial_\mu \phi^a)^2 f_{aa}(\phi^b) = \mathcal{G}'_{aa}, \quad \forall \mu, \quad (3.15)$$

$$\sum_{a=0}^3 \partial_\mu \phi^a \partial_\nu \phi^a f_{aa}(\phi^b) = 0, \quad \text{when } \mu \neq \nu.$$

Note that if we propose the condition

$$\partial_\mu \phi^a \partial_\nu \phi^a = 0, \quad \forall a, \quad \text{and } \mu \neq \nu, \quad (3.16)$$

then we can satisfy the second set of equations in (3.15). Furthermore these equations namely (3.16) can be solved by demanding

$$\partial_{\mu \neq a} \phi^a = 0, \quad \forall \mu, a, \quad (3.17)$$

meaning that $\phi^a = \phi^a(x^a)$ only. On the other hand as discussed in [10, 11] when (3.16) is used in the first set of equations in (3.15) one can simplify the first set to the form

$$(\partial_c \phi^c)^2 f_{cc} = \mathcal{G}'_{cc}, \quad \forall c, \quad \text{no sum on } c. \quad (3.18)$$

By taking square root on both sides these equations become

$$\partial_c \phi^c \sqrt{f_{cc}} = \pm \sqrt{\mathcal{G}'_{cc}}. \quad (3.19)$$

Finally if we choose the diagonal components of the background metric \bar{f} as

$$f_{cc} = \frac{\mathcal{G}'_{cc}}{(F_c(x^c))^2}, \quad (3.20)$$

so that

$$\bar{f} = \begin{pmatrix} \frac{\mathcal{G}'_{00}}{(F_0(x^0))^2} & 0 & 0 & 0 \\ 0 & \frac{\mathcal{G}'_{11}}{(F_1(x^1))^2} & 0 & 0 \\ 0 & 0 & \frac{\mathcal{G}'_{22}}{(F_2(x^2))^2} & 0 \\ 0 & 0 & 0 & \frac{\mathcal{G}'_{33}}{(F_3(x^3))^2} \end{pmatrix}, \quad (3.21)$$

then

$$\phi^c(x^c) = \pm \int F_c(x^c) dx^c, \quad (3.22)$$

become the solutions of (3.18). Here we have introduced the completely arbitrary integrable functions $F_a(x^a)$'s. These Stückelberg scalar field solutions when the background metric is chosen as (3.21) are the solutions of the ansatz (2.10). We should also state that to be able to construct (3.21) explicitly one first has to specify the effective energy-momentum tensor \tilde{T} (which has to obey a conservation equation as we will discuss below) then one has to solve the diagonal metric from the metric sector. We see that the form of our solutions justifies the diagonal matrix assumption we have made for (2.10) all through our analysis. On the other hand for more general non-diagonal forms of $\sqrt{\Sigma}$, g , and \tilde{T} one would have a mixed-differential term-wise coupling in the partial differential equations (3.13) which can not be so easily brought to a decoupled form. Those equations may also admit solutions in principle however the reader should appreciate that solving them would be more involved than the simpler decoupling solution method we have discussed above. Another essential loss of generality was choosing the background metric diagonal for decoupling and consequently generating solutions to these partial differential equations. Assuming non-diagonal background metric forms would cause similar coupling complications. We further note that all the conditions on Δ needed to obtain the real roots of (3.4) contain the matrix D in them. For this reason in general the sign conditions on Δ for the reality of the solutions (as Δ is a functional of \tilde{T} via D) depending on the particular choice of \tilde{T} may or may not bring restrictions on its domain. Similarly as $\sqrt{\Sigma}$ is a functional of \tilde{T} too the valid domain of the solutions of the background metric and the Stückelberg fields may also be restricted for certain class of solutions of (3.4) and the choice of \tilde{T} . However as we have mentioned before there also exists the freedom of assigning $e_1 = tr\sqrt{\Sigma}$ as a compensation in the most general set-up to design solutions which are free of these restrictions. Now on the other hand if we set

$$B^2 - 3AC = 0, \quad (3.23)$$

which puts no restrictions on the regions of validity of the solutions and the effective source³ then it is guaranteed that there is one real matrix solution

³ Eq. (3.23) does not result in the above mentioned restrictions on the domain of validity of solutions as Δ_0 does not depend on \tilde{T} .

to (3.4) and it is

$$\sqrt{\Sigma} = -\frac{1}{3}A^{-1}(B + (\Delta_1)^{1/3}), \quad (3.24)$$

where due to the condition (3.23) we have

$$\Delta_1 = -B^3 + 27A^2D. \quad (3.25)$$

We remark that since all the matrices in the above relations are diagonal the power operations can directly be applied on the diagonal entries. Now unlike the more general solutions we have discussed the condition (3.23) will determine e_1 . To see this first note that from (3.23) we get

$$e_2 = \frac{1}{3}\left(e_1^2 - \frac{\beta_2}{\beta_3}e_1 + \frac{\beta_2^2}{\beta_3^2} - \frac{3\beta_1}{\beta_3}\right). \quad (3.26)$$

Substituting (3.3) into this equation we obtain

$$\beta_2 e_1^2 + \left(6\beta_1 - \frac{\beta_2^2}{\beta_3}\right)e_1 + \frac{\beta_2^3}{\beta_3^2} - \frac{3\beta_1\beta_2}{\beta_3} - 9C_1 + 9\beta_0 + \frac{3}{2}C_2\tilde{T}^\mu{}_\mu - 3C_2\tilde{\tau}^\mu{}_\mu = 0, \quad (3.27)$$

which is a quadratic equation for $e_1 = \text{tr}\sqrt{\Sigma}$. The discriminant of this equation is

$$\Delta' = 36C_1\beta_2 + 36\beta_1^2 - \frac{3\beta_2^4}{\beta_3^2} - 36\beta_0\beta_2 - 6C_2\beta_2\tilde{T}^\mu{}_\mu + 12C_2\beta_2\tilde{\tau}^\mu{}_\mu. \quad (3.28)$$

In order to have a real root for (3.27) we must have

$$\Delta' \geq 0. \quad (3.29)$$

This brings a constraint on the trace of the effective energy-momentum tensor \tilde{T} as well as the one for $\tilde{\tau}$. We will see in the cosmological solution scheme of the next section that choosing equality in (3.29) will fix the equation of state of the effective ideal fluid. In spite of this restriction on the other hand this solution has physical advantages as it will not cause additional constraints on the building blocks of the cosmological solutions. However if equality is not chosen close inspection shows that the free parameters in (3.28) can still be tuned to give solutions in physically sensible domains. For example to enlarge the domain of validity of the solutions when $C_1 > 0, \beta_2 > 0, C_2 < 0$ one can tune β_2 to small values and when $C_1 < 0, \beta_2 < 0, C_2 > 0$ one can

tune the norm of β_2 to high values to release the restrictions on \tilde{T}^μ_μ . On the other hand choosing the discriminant as zero will relax all the domain restrictions but fix the form of \tilde{T} in return. Now assuming (3.29) is satisfied, the real solutions to (3.27) become

$$e_1 = tr\sqrt{\Sigma} = -\frac{3\beta_1}{\beta_2} + \frac{\beta_2}{2\beta_3} \pm \left[\frac{9C_1}{\beta_2} + 9\left(\frac{\beta_1}{\beta_2}\right)^2 - \frac{3}{4}\left(\frac{\beta_2}{\beta_3}\right)^2 - 9\frac{\beta_0}{\beta_2} - \frac{3C_2}{2\beta_2}\tilde{T}^\mu_\mu + \frac{3C_2}{\beta_2}\tilde{\tau}^\mu_\mu \right]^{1/2}. \quad (3.30)$$

This result together with (3.3) fix the trace coefficients in the solution (3.24) of the ansatz equation (3.4). Now we can explicitly compute $\sqrt{\Sigma}$. Again from (3.24) we have

$$\sqrt{\Sigma} = \mathcal{H}, \quad (3.31)$$

where we define the matrix

$$\mathcal{H} = \frac{1}{3}\left(\frac{\beta_2}{\beta_3} + e_1\right)\mathbf{1}_4 - \frac{1}{3\beta_3}(\mathcal{F})^{1/3}, \quad (3.32)$$

with

$$\mathcal{F} = (\beta_2 + \beta_3 e_1)^3 \mathbf{1}_4 + 27\beta_3^2 (C_2 g^{-1} \tilde{T} + C_2 g^{-1} \tilde{\tau}). \quad (3.33)$$

Following the same track of solution route we have introduced earlier in this section we find that the background metric \bar{f} becomes

$$\bar{f} = diag\left(\frac{\mathcal{H}'_{00}}{(F_0(x^0))^2}, \frac{\mathcal{H}'_{ii}}{(F_i(x^i))^2}\right), \quad (3.34)$$

where $\mathcal{H}' = g\mathcal{H}^2$. We can also write down the Stückelberg scalar solutions again as

$$\phi^a(x^a) = \pm \int F_a(x^a) dx^a. \quad (3.35)$$

Although by now we can explicitly construct the scalar solutions for the more general case in (3.22) or the special one in (3.35) their being the solutions of the Stückelberg sector of (2.1) is not guaranteed yet. To guarantee this we must focus on the scalar field equations of (2.1). It can be directly deduced from the metric equation (2.4) that the scalar field equations must be equivalent to the covariant constancy condition

$$\nabla^\mu \mathcal{T}_{\mu\nu}^S = 0, \quad (3.36)$$

where ∇^μ is the covariant derivative of the Levi-Civita connection of g . If we substitute our ansatz (2.10) in this equation we get

$$\nabla^\mu \mathcal{T}_{\mu\nu}^S = \nabla^\mu [C_1 m^2 g_{\mu\nu} + C_2 m^2 \tilde{T}_{\mu\nu} + C_2 m^2 \tilde{\tau}_{\mu\nu} - \frac{1}{2} C_2 m^2 \tilde{T}^\rho{}_\rho g_{\mu\nu}] = 0. \quad (3.37)$$

Then by using the metric compatibility $\nabla^\mu g_{\alpha\beta} = 0$ we obtain a constraint

$$\nabla^\mu [\tilde{T}_{\mu\nu} + \tilde{\tau}_{\mu\nu} - \frac{1}{2} \tilde{T}^\rho{}_\rho g_{\mu\nu}] = 0, \quad (3.38)$$

which can be considered as a modified conservation or continuity equation for the effective energy-momentum tensor \tilde{T} . Thus finally we conclude that if one chooses \tilde{T} in (2.10) as a solution of (3.38) then for the background metric (3.21) or (3.34) the Stückelberg scalar solutions (3.22) or (3.35) of (2.10) respectively become the scalar field solutions of the massive gravity action (2.1) together with the diagonal metric g to be solved from the metric sector which we will inspect for cosmological cases next.

4 FLRW Dynamics

Now we turn our attention to the metric sector. If we substitute the ansatz (2.10) into the metric equation (2.4) we get the on-shell equation

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} - \tilde{\Lambda} g_{\mu\nu} + C_2 m^2 \tilde{T}_{\mu\nu} + C_2 m^2 \tilde{\tau}_{\mu\nu} - \frac{1}{2} C_2 m^2 \tilde{T}^\rho{}_\rho g_{\mu\nu} = G_N T_{\mu\nu}^M, \quad (4.1)$$

where we have defined the effective cosmological constant

$$\tilde{\Lambda} = \frac{1}{2} \Lambda - C_1 m^2. \quad (4.2)$$

We see that upon using the ansatz (2.10) the metric sector is completely decoupled from the scalars whose contribution is truncated to the presence of an effective cosmological constant and an energy-momentum tensor. Let us consider (4.1) for the cosmological FLRW metric

$$g = -dt^2 + \frac{a^2(t)}{1 - kr^2} dr^2 + a^2(t) r^2 d\theta^2 + a^2(t) r^2 \sin^2 \theta d\varphi^2, \quad (4.3)$$

in the spatially spherical coordinates $\{t, r, \theta, \varphi\}$. We should note at this point that the cosmological metric is diagonal in this frame so that our analysis

in the previous section is applicable. For consistency with the isotropy and homogeneity in (4.3) we also choose the physical and the effective sources in (4.1) as ideal fluids so that

$$T_{\mu\nu}^M = (\rho(t) + p(t))U_\mu U_\nu + p(t)g_{\mu\nu}, \quad (4.4)$$

$$\tilde{T}_{\mu\nu} = (\tilde{\rho}(t) + \tilde{p}(t))U_\mu U_\nu + \tilde{p}(t)g_{\mu\nu}.$$

From these definitions we can deduce the trace of T^M and \tilde{T} as

$$T_\mu^{M\mu} = g^{\mu\nu}T_{\mu\nu}^M = 3p - \rho, \quad (4.5)$$

$$\tilde{T}^\mu_\mu = g^{\mu\nu}\tilde{T}_{\mu\nu} = 3\tilde{p} - \tilde{\rho}.$$

Also referring to its definition in (2.12) via (4.4) $\tilde{\tau}$ can be computed as

$$\tilde{\tau}_{\mu\nu} = -\tilde{p}g_{\mu\nu}, \quad (4.6)$$

for which we have taken $\tilde{\rho}, \tilde{p}$ to be linearly independent with $g^{\mu\nu}$. Its trace becomes

$$\tilde{\tau}^\mu_\mu = -4\tilde{p}. \quad (4.7)$$

If now we use the metric (4.3) in (4.1) a standard computation which is slightly modified due to the extra terms in (4.1) gives the tt -component equation

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{G_N}{3}\rho - \frac{C_2 m^2}{6}(5\tilde{p} + \tilde{\rho}) - \frac{\tilde{\Lambda}}{3}, \quad (4.8)$$

which is the modified Friedmann equation. The three spatial component ii -equations lead to the same form of equation

$$\frac{2\ddot{a}}{a} = -\left(\frac{\dot{a}}{a}\right)^2 - \frac{k}{a^2} - G_N p - \frac{C_2 m^2}{2}(3\tilde{p} - \tilde{\rho}) - \tilde{\Lambda}. \quad (4.9)$$

By using (4.8) in (4.9) we obtain the modified cosmic acceleration equation

$$\frac{\ddot{a}}{a} = -\frac{G_N}{6}(3p + \rho) + \frac{C_2 m^2}{3}(\tilde{\rho} - \tilde{p}) - \frac{\tilde{\Lambda}}{3}. \quad (4.10)$$

For the FLRW metric (4.3) the energy-momentum conservation law

$$\nabla^\mu T_{\mu\nu}^M = 0, \quad (4.11)$$

of the physical matter leads to the continuity equation

$$\dot{\rho} = -\frac{3\dot{a}}{a}(p + \rho). \quad (4.12)$$

On the other hand if we compute the modified conservation equation (3.38) for the effective energy-momentum tensor \tilde{T} of the ideal fluid defined in (4.4) by using the FLRW metric (4.3) then for the spatial components $\nu = i = 1, 2, 3$ we get the usual null result. However the t -component of (3.38) yields

$$\frac{1}{2}(5\dot{\tilde{p}} + \dot{\tilde{\rho}}) = -\frac{3\dot{a}}{a}(\tilde{p} + \tilde{\rho}). \quad (4.13)$$

Therefore like the one for the physical matter this modified conservation equation couples our ansatz-originated effective ideal fluid to the Friedmann equations. As the reader may realize this continuity equation is different than the physical one (4.12). The main reason for this is that we demand the effective ideal fluid energy-momentum tensor to emerge from the Lagrangian (2.17) rather than the usual physical Lagrangian of the perfect fluids which would be simply the effective pressure and in which the first law of thermodynamics is implicitly casted. For this reason the effective ideal fluids taking role in the cosmological dynamics here can be called non-physical. Specifying the equation of state of the physical and the effective matter will enable us to solve for the scale factor $a(t)$, the physical and the effective pressure and the energy densities. The solved exact form of the effective energy-momentum tensor together with the FLRW metric (4.3) then can be used in the results of the previous section to explicitly construct the background metric \bar{f} which generates these solutions. On the other hand, in our solution scheme of the scalar sector presented in Section three the Stückelberg scalars and the effective source namely, in general \tilde{T} or in special for the cosmological case the components $\{\tilde{\rho}, \tilde{p}\}$ of it are independent from each other. For the special class of diagonal solutions we have constructed one first chooses the functions $\{F_a(x^a)\}$'s, then determines the Stückelberg scalars from (3.22) by integrating these functions. After deciding the form of \tilde{T} (i.e introducing the ideal effective fluid arguments $\{\tilde{\rho}, \tilde{p}\}$ for the cosmological case) one later solves the physical metric and $\tilde{T}/\{\tilde{\rho}, \tilde{p}\}$ from the metric, the physical matter equations, and the conservation equation of \tilde{T} (which corresponds to the scalar field equations as we have discussed in the previous section). Finally, one uses these solutions namely, g , and $\tilde{T}/\{\tilde{\rho}, \tilde{p}\}$ together with the functions $\{F_a(x^a)\}$ to construct the necessary background metric \bar{f} in (3.21) which consistently

leads to the derived solutions both in the Stückelberg and the metric sectors of the theory. Therefore we see that in this branch of diagonal solutions for the cosmological case the Stückelberg scalars and the effective fluid properties $\{\tilde{\rho}, \tilde{p}\}$ are completely independent of each other as the formers are arbitrary. On the other hand, in the more general picture of solutions (at least for the ones again diagonal in $\{\sqrt{\Sigma}, g, \bar{f}\}$) one may follow a different but a more challenging track to generate a broader class of solutions. In this method for example for the cosmological solutions one can first solve $\{\tilde{\rho}, \tilde{p}\}$ (in other general cases, the components of \tilde{T}) from the Friedmann equations (modified Einstein equations), the physical matter equations, and the conservation equation of \tilde{T} , and now one can independently choose a background metric \bar{f} , then one can solve the coupled partial differential equations in (3.13) to obtain the Stückelberg scalars in terms of $\{g, \bar{f}, \tilde{\rho}, \tilde{p}\}$. Hence, in this more general solution scenario which we have not considered here the Stückelberg scalars become functions of the thermodynamic state of the effective ideal fluid. We should also remark another important point here. Although the equation of state of the physical matter is subject to natural constraints we are completely free to choose any form $\tilde{p} = \tilde{p}(\tilde{\rho})$ of it for the effective matter case. Even non-physical effective ideal fluid choices are possible provided they satisfy (4.13) which is different than the universal energy-momentum conservation law (4.12) of physical matter. However as we have discussed in the previous section in spite of this large freedom we have bounds on the effective energy-momentum tensor to have real solutions of the reference metric and the scalar sector. On the other hand in the special solution case if (3.29) is saturated that is to say if $\Delta' = 0$ then the solutions are valid for the entire coordinate span but now we have to fix the equation of state of the effective matter as

$$\tilde{p} = \frac{1}{11}\tilde{\rho} + C', \quad (4.14)$$

where

$$C' = \frac{1}{11} \left[\frac{6C_1}{C_2} + \frac{6\beta_1^2}{C_2\beta_2} - \frac{\beta_2^3}{2C_2\beta_3^2} - \frac{6\beta_0}{C_2} \right]. \quad (4.15)$$

We have obtained (4.14) by using (4.5) and (4.7) in (3.28) then by equating the result to zero. Another way of obtaining real solutions in (3.27) is to equate the constant coefficient to zero. This leads to an equation of state

$$\tilde{p} = \frac{1}{11}\tilde{\rho} + C'', \quad (4.16)$$

with

$$C'' = \frac{2}{33C_2} \left[9C_1 - 9\beta_0 - \frac{\beta_2^3}{\beta_3^2} + \frac{3\beta_1\beta_2}{\beta_3} \right], \quad (4.17)$$

where we have used (4.5) and (4.7). Now before we conclude let us turn attention to the Stückelberg sector solutions of the special type which satisfy (3.23) and which accompany the cosmological metric solutions we have discussed here. We have from (4.4)

$$\tilde{T} = \begin{pmatrix} \tilde{\rho} & 0 & 0 & 0 \\ 0 & \tilde{p}g_{11} & 0 & 0 \\ 0 & 0 & \tilde{p}g_{22} & 0 \\ 0 & 0 & 0 & \tilde{p}g_{33} \end{pmatrix}, \quad (4.18)$$

for the coordinate system defined in (4.3). So we get

$$g^{-1}\tilde{T} = \begin{pmatrix} -\tilde{\rho} & 0 & 0 & 0 \\ 0 & \tilde{p} & 0 & 0 \\ 0 & 0 & \tilde{p} & 0 \\ 0 & 0 & 0 & \tilde{p} \end{pmatrix}. \quad (4.19)$$

Also via (4.6)

$$g^{-1}\tilde{\tau} = \begin{pmatrix} -\tilde{p} & 0 & 0 & 0 \\ 0 & -\tilde{p} & 0 & 0 \\ 0 & 0 & -\tilde{p} & 0 \\ 0 & 0 & 0 & -\tilde{p} \end{pmatrix}. \quad (4.20)$$

Therefore by inspecting (3.32) and (3.33) under these identifications we find that

$$\mathcal{H}' = \begin{pmatrix} \mathcal{H}'_{00} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (4.21)$$

where

$$\mathcal{H}'_{00} = -\frac{1}{9\beta_3^2} \left[\beta_2 + \beta_3 e_1 + \left[-(\beta_2 + \beta_3 e_1)^3 + 27\beta_3^2 C_2 (\tilde{p} + \tilde{\rho}) \right]^{1/3} \right]^2. \quad (4.22)$$

Thus from (3.34) we conclude that the background metric \bar{f} is also of the form (4.21) and it reads

$$\bar{f} = \text{diag}\left(\frac{\mathcal{H}'_{00}}{(F_0(x^0))^2}, 0, 0, 0\right). \quad (4.23)$$

Now if we analyze (3.19) for our special solution satisfying (3.23) we observe that within the cosmological solutions we have discussed in this section three of the Stückelberg scalars decouple from the metric sector and become completely arbitrary with no role in the effective terms in the metric equation. Thus in these special type of solutions the construction of \mathcal{T}^S in (2.9) which enters into the metric equation (2.4) via the ansatz (2.10) it satisfies gets no contribution from ϕ^i 's but only from ϕ^0 for generating the cosmological solutions of this section. However this trivial solution set which is legitimate analytically does not promise a physically acceptable case nor the background metric (4.23). For this reason in this special case for the Stückelberg sector instead of choosing the solutions of (3.19) as in (3.20) which determine the background metric one should follow a different method in which one should choose the components of the background metric completely or partially and then solve for the scalars. For example if one chooses

$$\bar{f} = \text{diag}(\frac{\mathcal{H}'_{00}}{(F_0(x^0))^2}, f_{11}(x^\mu), f_{22}(x^\mu), f_{33}(x^\mu)), \quad (4.24)$$

where $\{f_{ii}(x^\mu)\}$ are arbitrary and

$$\phi^0(x^0) = \pm \int F_0(x^0) dx^0, \quad \phi^1, \phi^2, \phi^3 = \text{constant}, \quad (4.25)$$

then one can still satisfy (3.19). On the other hand beside the above mentioned somewhat trivial cases we can find more general solutions of $\{\phi^a\}$ for a completely specified background metric by solving the equations in (3.15) where one should replace \mathcal{G}' by (4.21). Finally we should remark at this point that for the more general case of background metric and the Stückelberg scalar solutions defined via the equations (3.7)-(3.22) there appears no problem of triviality occurring in the form of both \bar{f} and $\{\phi^a\}$'s like the one we have encountered above for the solutions satisfying the condition (3.23).

5 Conclusion

We have constructed a specially chosen solution ansatz which has enabled us to devise a method to find solutions to the Stückelberg scalar sector of the massive gravity theory. We have done this by decoupling the scalar sector from the metric one and by reducing the task to finding the solutions of a mutual constraint equation of the background metric and the scalars of

the theory. In this way we were able to find a broad solution class of the Stückelberg sector by specifying the associated background metric in terms of the physical one and the ansatz parametrizing effective matter energy-momentum tensor. In this solution scheme the metric sector only sees the scalars as a collective effect in the presence of an effective energy-momentum tensor whose conservation equation is modified by a variation and a trace term. Later we constructed the associated modified FLRW cosmological dynamics by assigning an ideal fluid nature to both the ordinary and the effective matter. We should emphasize the point here that although the equations of different sectors are decoupled from each other the solutions are intimately related. One should first solve the physical metric and the effective energy-momentum tensor from the metric and the modified conservation equations then one should use these solutions to construct the background metric which admits such a solution scheme. Finally the solutions of the Stückelberg scalars trivially follow this construction up to a set of integrable functions.

The cosmological solutions of the massive gravity is an active topic of research which has reached some positive and negative conclusions. For example it has been shown that for a flat background metric there exist open FLRW solutions [13] however there are no flat or closed FLRW solutions [14]. The open FLRW solutions with the flat background metric have stability problems [15, 16]. These facts led to the idea of choosing different background metrics for the cosmological solutions such as the de Sitter [17] and the FLRW type [18] ones. However these solutions have non-physical consequences regarding the Higuchi bound. On the other hand the inhomogeneous and anisotropic cosmological solutions of the ghost-free massive gravity have also been studied giving physically sensible results in certain regions of the parameter space of the theory [19, 20]. Contrary to the mainstream of the corresponding literature we should remark that our approach does not pre-determine the background metric but solves it for the particular physical scenario in inspection. Therefore the formalism of the present work addresses not only to a very rich sector of the solution moduli of the background metric and the scalars of the theory but also to a wide range of physical applications via the choice of the effective matter. We certainly owe this richness to the large solution space of the massive gravity. We hope that the solution methodology achieved here will provide an appropriate framework to generate new and physically acceptable cosmological or astrophysical solutions of the theory. However in this direction both the stability issues and

the Higuchi bound behavior of the solutions found here need to be inspected in detail elsewhere.

We observe that the effective and collective contribution of the scalars and the reference metric to the FLRW dynamics is suppressed by a squared graviton mass coefficient. However there also appear the C_1 and C_2 factors which possibly contain the degree of freedom of tuning the solutions for agreement with the physical needs. Such a scale generation may physically justify our construction as an effective theory of sub-solutions of the the exact theory. Both in the more general (which admits a wide range of solutions depending on specifying the effective ideal fluid) and the special cases we have mentioned, the modified cosmological dynamics can separately be studied and their testable consequences can further be classified as well. In this direction we should remark that the solutions we have constructed do contain self-accelerating ones. In particular, when we set C_2 to zero then we have the solutions in which the contribution of the mass sector to the metric one becomes solely an effective cosmological constant (where still the richness of the corresponding Stückelberg scalar and the background metric solutions remains intact) which is similar to many of such solutions constructed in the literature. However the more interesting class of self-accelerating solutions can be obtained by allowing the existence of effective fluids. The reader may appreciate the possibility of a wide range of such solutions as a consequence of the free choice of the state equation for these fluids which in general is completely arbitrary. For this reason we leave a detailed examination of this issue for a later work. More generally the point which deserves to be mentioned, and emphasized on, is that, the present construction enables the freedom of choosing any ordinary or exotic form of effective matter which will admit a non-physical dynamics in generating solutions. This is a consequence of the fact that although we have specified the effective energy-momentum tensor in the ideal fluid form, while we construct the on-shell Lagrangian corresponding to the solution ansatz we did not take the usual Lagrangian of the ideal fluid which would be just the effective pressure upon using the first law of thermodynamics and which would generate the perfect fluid energy-momentum tensor with no extra terms in the metric equation. Thus our choice of the effective fluid does not obey the first law of thermodynamics as can be obviously seen also from the modified energy-momentum relation it satisfies and for this reason it can be called non-physical. This brings the opportunity of generating a rich class of solutions which would especially emerge from effective fluid choices which can not have physical

correspondents. Furthermore the existence of the implicit solution relation we have discussed above between the physical, the background metrics and the effective matter also suggests a coupling dynamics among them. Such a construction which could include a dynamical nature for the background metric and a reasonable origin for the effective matter can be searched within the context of bi-metric gravity cosmological solutions [21, 22, 23, 24]. One can also separately consider to extend the ansatz we have used to generate similar solutions of bigravity. Finally, we should point out the possibility of modifying our ansatz to other forms which may lead to various other solutions not necessarily cosmological within the same line of reasoning.

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